

FINITE TIME EXTINCTION FOR SOLUTIONS TO FAST DIFFUSION STOCHASTIC POROUS MEDIA EQUATIONS

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ABSTRACT. *We prove that the solutions to fast diffusion stochastic porous media equations have finite time extinction with strictly positive probability.*

1. INTRODUCTION

Consider the stochastic porous media equation

$$(1) \quad \begin{cases} dX(t) - \rho \Delta(|X|^\alpha(t) \operatorname{sign} X(t))dt - \Delta(\tilde{\Psi}(X(t)))dt = \sigma(X(t))dW(t), & \text{in } (0, \infty) \times \mathcal{O}, \\ X = 0 & \text{on } (0, \infty) \times \partial\mathcal{O}, \quad X(0, x) = x & \text{on } \mathcal{O}, \end{cases}$$

where $\rho > 0$, $\alpha \in (0, 1)$, $\tilde{\Psi}$ is a continuous monotonically non decreasing function of linear growth and $\sigma(X)dW = \sum_{k=1}^{\infty} \mu_k X e_k d\beta_k$, $t \geq 0$, where $\{\beta_k\}$ is a sequence of independent real Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ and $\{e_k\}$ is an orthonormal basis in $L^2(\mathcal{O})$ which for convenience will be taken as the eigenfunction system for the Laplace operator with Dirichlet boundary conditions, i.e., $-\Delta e_k = \lambda_k e_k$ in \mathcal{O} , $e_k = 0$ on $\partial\mathcal{O}$, where \mathcal{O} is an open and bounded subset of \mathbb{R}^d , with smooth boundary $\partial\mathcal{O}$. We shall assume that $\sum_{k=1}^{\infty} \mu_k^2 \lambda_k^2 < \infty$. Equation (1) for $0 < \alpha < 1$ is relevant in the mathematical modelling of the dynamics of an ideal gas in a porous medium and, in particular, in a plasma fast diffusion model (for $\alpha = 1/2$) (see e.g. [4]). The existence and uniqueness of a strong solution in the sense to be defined below was studied in [1],[2],[3],[5] for more general nonlinear stochastic equations of the form (1). In [3] (see also [1]) it was also proven that for $\alpha = 0$ and $d = 1$ the solution $X = X(t, x)$ to (1) has the finite extinction property: $\mathbb{P}(\tau \leq n) \geq 1 - \frac{|x|_{-1}}{\rho\gamma} \left(\int_0^n e^{-C_N s} ds \right)^{-1}$ for $|x|_{-1} < C_N^{-1} \rho\gamma$ where $\tau = \inf\{t \geq 0 : |X(t, x)|_{-1} = 0\} = \sup\{t \geq 0 : |X(t, x)|_{-1} > 0\}$ and C_N, γ are constants related to the Wiener process W and respectively to the domain $\mathcal{O} \subset \mathbb{R}^1$.

The following notations will be used in the sequel. $H = L^2(\mathcal{O})$, $p \geq 1$, with the norm denoted by $|\cdot|_2$ and scalar product $\langle \cdot, \cdot \rangle$. $H^{-1}(\mathcal{O})$ is the dual of the Sobolev space $H_0^1(\mathcal{O})$ and is endowed with the scalar product $\langle u, v \rangle_{-1} = \langle u, (-\Delta)^{-1}v \rangle$, where Δ is the Laplace operator with domain $H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$. All processes $X = X(t)$ arising here are adapted with respect to the filtration $\{\mathcal{F}_t\}$. For a Banach space E , $L_W^p(0, T; E)$ denotes the space of all adapted processes in $L^p(0, T; E)$. We shall use standard notation for Sobolev spaces and spaces of integrable functions on \mathcal{O} .

2. THE MAIN RESULT

Definition 2.1. Let $x \in H$. An H -valued continuous (\mathcal{F}_t) -adapted process $X = X(t, x)$ is called a solution to (1) on $[0, T]$ if $X \in L^p(\Omega \times (0, T) \times \mathcal{O}) \cap L^2(0, T; L^2(\Omega, H))$, $p \geq 2$, such that \mathbb{P} -a.s. $\forall j \in \mathbb{N}$, $t \in [0, T]$,

$$(2) \quad \begin{aligned} \langle X(t, x), e_j \rangle &= \langle x, e_j \rangle + \int_0^t \int_{\mathcal{O}} (\rho |X(s, x)(\xi)|^\alpha \operatorname{sign} X(s, x)(\xi) + \tilde{\Psi}(X(s, x)(\xi))) \Delta e_j(\xi) d\xi ds \\ &\quad + \sum_{k=1}^\infty \mu_k \int_0^t \langle X(s, x) e_k, e_j \rangle d\beta_k(s), \end{aligned}$$

For $x \in L^p(\mathcal{O})$, $p \geq 4$ and $d = 1, 2, 3$ there is a unique solution $X \in L_W^\infty(0, T; L^p(\Omega, H))$ to (1) in the sense of Definition 2.1. Moreover, if $x \geq 0$ a.e. in \mathcal{O} then $X \geq 0$ a.e. in $\Omega \times [0, T] \times \mathcal{O}$.

By the proof of [3, Theorem 2.2] and [3, Proposition 3.4] we also know that for $\lambda \rightarrow 0$,

$$(3) \quad \begin{cases} X_\lambda \rightarrow X & \text{strongly both in } L^2(0, T; L^2(\Omega, L^2(\mathcal{O}))) \text{ and in } L^2(\Omega; C([0, T]; H)), \\ & \text{weakly in } L^p(\Omega \times (0, T) \times \mathcal{O}), \text{ and weak}^* \text{ in } L^\infty(0, T; L^p(\Omega; L^p(\mathcal{O}))), \end{cases}$$

where X_λ , $\lambda > 0$, is the solution to approximating equation

$$(4) \quad \begin{cases} dX_\lambda(t) - \Delta(\Psi_\lambda(X_\lambda(t)) + \lambda X_\lambda(t) + \tilde{\Psi}(X_\lambda(t)))dt = \sigma(X_\lambda(t))dW(t), \\ \Psi_\lambda(X_\lambda) + \lambda X_\lambda + \tilde{\Psi}(X_\lambda) = 0 \quad \text{on } \partial\mathcal{O}, \quad X_\lambda(0, x) = x, \\ \Psi_\lambda(x) = \frac{1}{\lambda} (x - (1 + \lambda\Psi_0)^{-1}(x)) = \Psi_0((1 + \lambda\Psi_0)^{-1}(x)), \quad \Psi_0(x) = \rho|x|^\alpha \operatorname{sign} x. \end{cases}$$

Everywhere in the sequel $X = X(t, x)$ is the solution to (1) in the sense of Definition 2.1 where $x \in L^4(\mathcal{O})$. Below γ shall denote the minimal constant arising in the Sobolev embedding $L^{\alpha+1}(\mathcal{O}) \subset H^{-1}(\mathcal{O})$ (see (7) below) and $C^* = \sum_{k=1}^\infty \mu_k^2 |e_k|_{H_0^1(\mathcal{O})}^2 = \sum_{k=1}^\infty \mu_k^2 \lambda_k^2$. Theorem 2.2 is the main result of the paper.

Theorem 2.2. Assume that $d = 1, 2, 3$ and that $0 < \alpha < 1$ if $d = 1, 2$, $\frac{1}{5} \leq \alpha < 1$ if $d = 3$. Let $\tau := \inf\{t \geq 0 : |X(t, x)|_{-1} = 0\}$. Then we have $|X(t, x)|_{-1} = 0$, for $t \geq \tau$, \mathbb{P} -a.s.. Furthermore $\mathbb{P}(\tau \leq t) \geq 1 - \frac{|x|_{-1}^{1-\alpha}}{(1-\alpha)\rho\gamma^{1+\alpha}} \left(\int_0^t e^{-(1-\alpha)C^*s} ds \right)^{-1}$. In particular, if $|x|_{-1}^{1-\alpha} < \frac{\rho\gamma^{1+\alpha}}{C^*}$, then $\mathbb{P}(\tau < \infty) > 0$, and if $C^* = 0$, then $\tau \leq \frac{|x|_{-1}^{1-\alpha}}{(1-\alpha)\rho\gamma^{1+\alpha}}$.

Remark 1. This result extends to $\mathcal{O} \subset \mathbb{R}^d$ with $d \geq 4$, if $\alpha \in [\frac{d-2}{d+2}, 1)$. However, we have to strengthen the assumption on μ_k , $k \in \mathbb{N}$, see [1, Section 4] and in particular [6, Remark 2.9(iii)] for a detailed discussion.

3. Proof of Theorem 2.2

We shall proceed as in the proof of [3, Theorem 4.2]. Consider the solution $X_\lambda \in L_W^2(0, T; L^2(\Omega; H_0^1(\mathcal{O})))$ to equation (4). Then by applying the classical Itô formula to the real valued semi-martingale $|X_\lambda(t)|_{-1}^2, t \in [0, T]$, and to the function

$\varphi_\varepsilon(r) = (r + \varepsilon^2)^{(1-\alpha)/2}$, $r \in \mathbb{R}$, we find that

$$\begin{aligned}
(5) \quad & d\varphi_\varepsilon(|X_\lambda(t)|_{-1}^2) + (1-\alpha)(|X_\lambda(t)|_{-1}^2 + \varepsilon^2)^{-(1+\alpha)/2} \langle X_\lambda(t), \Psi_\lambda(X_\lambda(t)) + \lambda X_\lambda(t) + \tilde{\Psi}_\lambda(X_\lambda(t)) \rangle dt \\
&= \frac{1}{2} \sum_{k=1}^{\infty} \mu_k^2 (1-\alpha) \frac{|X_\lambda(t)e_k|_{-1}^2 (|X_\lambda(t)|_{-1}^2 + \varepsilon^2)^{-(1+\alpha)/2} - (1-\alpha)^2 |\langle X_\lambda(t)e_k, X_\lambda(t) \rangle_{-1}|^2}{(|X_\lambda(t)|_{-1}^2 + \varepsilon^2)^{(3+\alpha)/2}} dt \\
&+ \langle \sigma(X_\lambda(t)) dW(t), \varphi'_\varepsilon(|X_\lambda(t)|_{-1}^2) X_\lambda(t) \rangle_{-1} \\
&\leq \frac{1}{2} \sum_{k=1}^{\infty} \mu_k^2 \frac{(1-\alpha)|X_\lambda(t)e_k|_{-1}^2}{(|X_\lambda(t)|_{-1}^2 + \varepsilon^2)^{(1+\alpha)/2}} dt + \langle \sigma(X_\lambda(t)) dW(t), \varphi'_\varepsilon(|X_\lambda(t)|_{-1}^2) X_\lambda(t) \rangle_{-1} \\
&\leq C^* \frac{(1-\alpha)|X_\lambda(t)e_k|_{-1}^2}{(|X_\lambda(t)|_{-1}^2 + \varepsilon^2)^{(1+\alpha)/2}} dt + \langle \sigma(X_\lambda(t)) dW(t), \varphi'_\varepsilon(|X_\lambda(t)|_{-1}^2) X_\lambda(t) \rangle_{-1}
\end{aligned}$$

Then letting $\lambda \rightarrow 0$, by (3) we get that $\liminf_{\lambda \rightarrow 0} \int_0^T \langle \Psi_\lambda(X_\lambda(t)), X_\lambda(t) \rangle dt \geq \rho \int_0^T |X(t)|_{L^{1+\alpha}(\mathcal{O})}^{1+\alpha} dt$,

\mathbb{P} -a.s. and hence

$$\begin{aligned}
(6) \quad & \varphi_\varepsilon(|X(t)|_{-1}^2) + (1-\alpha)\rho \int_r^t \frac{|X(s)|_{L^{\alpha+1}(\mathcal{O})}^{\alpha+1}}{(|X(s)|_{-1}^2 + \varepsilon^2)^{(1+\alpha)/2}} ds \leq \varphi_\varepsilon(|X(r)|_{-1}^2) \\
&+ C^* \int_r^t \frac{(1-\alpha)|X(s)|_{-1}^2}{(|X(s)|_{-1}^2 + \varepsilon^2)^{(1+\alpha)/2}} ds + 2 \int_r^t \langle \sigma(X(s)) dW(s), \varphi'_\varepsilon(|X(s)|_{-1}^2) X(s) \rangle_{-1}, \quad \mathbb{P}\text{-a.s.}, \quad r < t.
\end{aligned}$$

Next by the Sobolev embedding theorem we have

$$\begin{aligned}
(7) \quad & |u|_{-1} \leq \gamma |u|_{L^{\alpha+1}(\mathcal{O})}, \quad \forall u \in L^{\alpha+1}(\mathcal{O}), \quad \text{if } d > 2 \text{ and } \alpha \geq \frac{d-2}{d+2}, \text{ and } \forall \alpha > 0, \text{ if } d=1,2.
\end{aligned}$$

Then substituting (7) into (6) we get

$$\begin{aligned}
(8) \quad & \varphi_\varepsilon(|X(t)|_{-1}^2) + (1-\alpha)\rho\gamma^{1+\alpha} \int_r^t \frac{|X(s)|_{-1}^{\alpha+1}}{(|X(s)|_{-1}^2 + \varepsilon^2)^{(1+\alpha)/2}} ds \leq \varphi_\varepsilon(|X(r)|_{-1}^2) \\
&+ C^* \int_r^t \frac{(1-\alpha)|X(s)|_{-1}^2}{(|X(s)|_{-1}^2 + \varepsilon^2)^{(1+\alpha)/2}} ds + \int_r^t \langle \sigma(X(s)) dW(s), \varphi'_\varepsilon(|X(s)|_{-1}^2) X(s) \rangle_{-1}, \quad \mathbb{P}\text{-a.s.}, \quad r < t.
\end{aligned}$$

Now for $\varepsilon \rightarrow 0$ we have

$$\begin{aligned}
& |X(t)|_{-1}^{1-\alpha} + (1-\alpha)\rho\gamma^{1+\alpha} \int_r^t 1_{\{|X(s)|_{-1} > 0\}} ds \leq |X(r)|_{-1}^{1-\alpha} + C^*(1-\alpha) \int_r^t |X(s)|_{-1}^{1-\alpha} ds \\
&+ (1-\alpha) \int_r^t \langle \sigma(X(s)) dW(s), |X(s)|_{-1}^{-(\alpha+1)} X(s) \rangle_{-1}, \quad \mathbb{P}\text{-a.s.}, \quad r < t.
\end{aligned}$$

Hence by Itô's product rule

$$\begin{aligned}
(9) \quad & e^{-C^*(1-\alpha)t} |X(t)|_{-1}^{1-\alpha} + (1-\alpha)\rho\gamma^{1+\alpha} \int_r^t e^{-C^*(1-\alpha)s} 1_{\{|X(s)|_{-1} > 0\}} ds \leq e^{-C^*(1-\alpha)r} |X(r)|_{-1}^{1-\alpha} \\
&+ (1-\alpha) \int_r^t e^{-C^*(1-\alpha)s} 1_{\{|X(s)|_{-1} > 0\}} \langle \sigma(X(s)) dW(s), |X(s)|_{-1}^{-(\alpha+1)} X(s) \rangle_{-1}, \quad \mathbb{P}\text{-a.s.}, \quad r < t.
\end{aligned}$$

From this it immediately follows that $e^{-C^*(1-\alpha)t} |X(t)|_{-1}^{1-\alpha}$, $t \geq 0$, is an (\mathcal{F}_t) -supermartingale, hence $|X(t)|_{-1} = 0$ for all $t \geq \tau$. So, (9) with $r = 0$ after taking expectation implies that $\int_0^t e^{-C^*(1-\alpha)s} \mathbb{P}(\tau > s) ds \leq \frac{|x|_{-1}^{1-\alpha}}{(1-\alpha)\rho\gamma^{1+\alpha}}$, $t \geq 0$. This implies that $\mathbb{P}(\tau > t) \leq \frac{|x|_{-1}^{1-\alpha}}{(1-\alpha)\rho\gamma^{1+\alpha}} \left(\int_0^t e^{-C^*(1-\alpha)s} ds \right)^{-1}$, $t \geq 0$, and the assertion follows. \square

ACKNOWLEDGEMENT

This work has been supported in part by the PIN-II ID-404 (2007-2010) project of Romanian Minister of Research, the DFG -International Graduate School “Stochastics and Real World Models”, the SFB-701 and the BiBoS-Research Center., the research programme “Equazioni di Kolmogorov” from the Italian “Ministero della Ricerca Scientifica e Tecnologica” and ”FCT, POCTI-219, FEDER”.

REFERENCES

- [1] V. Barbu, G. Da Prato and M. Röckner, *Existence and uniqueness of nonnegative solutions to the stochastic porous media equation*, Indiana University Math. Journal, **57**, 187-212, 2008.
- [2] V. Barbu, G. Da Prato and M. Röckner, *Existence of strong solutions for stochastic porous media equation under general monotonicity conditions*, Annals of Probability. (to appear).
- [3] V. Barbu, G. Da Prato and M. Röckner, *Stochastic porous media equations and self-organized criticality*, Comm. Math. Phys, (to appear).
- [4] J. Berryman and C. Holland, *Stability of the separable solution for fast diffusion*. Arch. Rational Mech. Anal. 74, no. 4, 379–388, 1980.
- [5] J. Ren, M. Röckner and F.Y. Wang, *Stochastic generalized porous media and fast diffusion equations*. J. Differential Equations 238, no. 1, 118–152, 2007.
- [6] M. Röckner and F.Y. Wang, *Non-monoton Stochastic generalized porous media equations*. Preprint, J. Differential Equations (to appear).

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